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Rationality of Finite Groups of Monomial Automorphisms of $k(x, y)$

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It has been shown (Hajja, *J. Algebra* **85** (1983), 243–250) that every finite cyclic group of monomial k-automorphisms of $k(x, y)$ has a purely transcendental fixed field. Here, the same result is established with the assumption that the group is cyclic dropped. © 1987 Academic Press, Inc.

Let K be a rational ($=$purely transcendental) extension of (the field) k of finite transcendence degree n and let G be a group of k-automorphisms of K. We say that G acts monomially iff for some base B (i.e., a transcendence basis of K over k for which $k(B) = K$), the subgroup $\langle k^*, B\rangle$ of K^* generated by k^* and B is stabilized by G. If $B = \{x_1, ..., x_n\}$ is a base of K on which G acts monomially, then to each $g \in G$ there corresponds $M = (M_{i,j}) \in GL(n, \mathbb{Z})$ in the manner that

$$g(x_i) \Big/ \prod_{j=1}^{n} x_j^{M_{j,i}} \in k^* \qquad \text{for } i = 1, 2, ..., n.$$

Let $p = p_B \colon G \to GL(n, \mathbb{Z})$ denote the map $g \to M$ and let G_0 denote its kernel. Note that if $B' = \{y_1, ..., y_n\}$ is another base of K consisting of elements in $\langle k^*, B\rangle$, then $p_{B'}(G) = P p_B(G) P^{-1}$ for some P in $GL(n, \mathbb{Z})$. Thus $p(G)$ is significant only up to conjugation. We now state and prove our result.

THEOREM. *Let $K = k(x, y)$ be a rational extension of k of transcendence degree 2 and let G be a finite group of monomial k-automorphisms of K. Then the subfield K^G of K fixed by G is rational (over k).*

Proof. By Fischer's argument [4], K^{G_0} is rational and G/G_0 acts on K^{G_0} as a group $\bar{G}$ of monomial k-automorphisms with $p \colon \bar{G} \to GL(2, \mathbb{Z})$ injective. Thus we may assume that $p \colon G \to GL(n, \mathbb{Z})$ is injective. We may further assume (in view of [2, Theorem 4]) that $p(G)$ is not cyclic. Thus up to

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conjugation, $p(G)$ is one of the following six groups [3, Chap. IX, 14, pp. 179–181],

$$G_1=\left\langle S_1=\begin{bmatrix}-1&0\\0&1\end{bmatrix}, S_2=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right\rangle,$$

$$G_2=\left\langle S_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}, S_2=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\right\rangle,$$

$$G_3=\left\langle S_1=\begin{bmatrix}0&-1\\-1&0\end{bmatrix}, S_2=\begin{bmatrix}0&-1\\1&-1\end{bmatrix}\right\rangle,$$

$$G_4=\left\langle S_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}, S_2=\begin{bmatrix}0&-1\\1&-1\end{bmatrix}\right\rangle,$$

$$G_5=\left\langle S_1=\begin{bmatrix}0&-1\\-1&0\end{bmatrix}, S_2=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right\rangle,$$

$$G_6=\left\langle S_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}, S_2=\begin{bmatrix}0&1\\-1&1\end{bmatrix}\right\rangle.$$

(These are taken from [3, p. 180] with $G_1=D_2$, $G_2=D_2$, $G_6=D_6$, $G_3=PD_3P^{-1}$, $G_4=PD_3P^{-1}$, $G_5=PD_4P^{-1}$, where $P=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$).

In each of these cases, we let s_1, s_2 be generators of G corresponding (under p) to the generators S_1, S_2 of $p(G)=G_i$. The rationality of K^G is now established for each case separately.

Case 1. If $p(G)=G_1$, then $G=\langle s_1, s_2\rangle$ acts on $K=k(x, y)$ as follows:

$$s_1:\quad x\to 1/c_1x, \qquad y\to u_1y,$$
$$s_2:\quad x\to u_2x, \qquad y\to 1/c_2y.$$

Since $\text{order}(s_1)=\text{order}(S_1)=2$, then $u_1^2=1$. Similarly $u_2^2=1$.

If u_1 or u_2 is 1 (say $u_1=1$), let $X=x+s_1(x)$. Then $K^{s_1}=k(X, y)$ and s_2 acts monomially on $k(X, y)$, namely

$$s_2:\quad X\to u_2X, \qquad y\to 1/c_2y.$$

Therefore $K^G=(K^{s_1})^{s_2}$ is rational by [2, Theorem 4].

Otherwise, $u_1=u_2=-1$ and $\text{char}(k)\neq 2$. Let $\bar{k}$ be the splitting field over k of $(T^2-c_1)(T^2-c_2)=(T+r_1)(T-r_1)(T+r_2)(T-r_2)$ and let $\bar{K}=\bar{k}(x, y)$. Let $\Gamma=\text{Gal}(\bar{k}/k)$. Then Γ is (a subgroup of the group) generated by

$$\gamma_1:\quad r_1\to -r_1, \qquad r_2\to r_2,$$
$$\gamma_2:\quad r_1\to r_1, \qquad r_2\to -r_2.$$

For any $t(\neq -1)$, let $t' = (1-t)/(1+t)$. Let

$$\begin{aligned} X &= r_1 x, & Y &= r_2 y, \\ A &= XY', & B &= X/Y', \\ U &= A'B', & W &= A'/B'. \end{aligned}$$

Then $\bar{K} = \bar{k}(x, y) = \bar{k}(X, Y) = \bar{k}(X, Y')$, and

$$s_2: \quad X \to -X, \qquad Y' \to -Y'.$$

Therefore $\bar{K}^{s_2} = \bar{k}(A, B) = \bar{k}(A', B')$, and

$$s_1: A' \to -A', \qquad B' \to -B'.$$

Hence $\bar{K}^G = (\bar{K}^{s_2})^{s_1} = \bar{k}(U, W) = \bar{k}(U', W')$. Now it is direct to check that Γ acts linearly on U', W'. Explicitly, γ_1 and γ_2 act as follows:

	X	Y	Y'	A	B	A'	B'	U	W	U'	W'
γ_1	$-X$	Y	Y'	$-A$	$-B$	$1/A'$	$1/B'$	$1/U$	$1/W$	$-U'$	$-W'$
γ_2	X	$-Y$	$1/Y'$	B	A	B'	A'	U	$1/W$	U'	$-W'$

Therefore K^G is rational by [1, Proposition 1.1].

Case 2. If $p(G) = G_2$, then one may clearly assume that G acts as follows:

$$\begin{aligned} &s_1: \quad x \to y \to x, \\ &s_2: \quad x \to 1/c_1 x, \qquad y \to 1/c_2 y. \end{aligned}$$

Since $s_1 s_2 = s_2 s_1$, then $c_1 = c_2 (= c$, say). Let $X = cxy$, $Y = x + y$. Then $K^{s_1} = k(X, Y)$ and s_2 acts monomially on $k(X, Y)$, namely

$$s_2: X \to 1/X, \qquad Y \to Y/X.$$

Hence K^G is rational by [2, Theorem 4].

Case 3. If $p(G) = G_3$, then one may assume that G acts as follows:

$$\begin{aligned} &s_1: x \to 1/y, \qquad y \to 1/x, \\ &s_2: x \to cy, \qquad y \to 1/dxy. \end{aligned}$$

Since $s_1 s_2 s_1 = s_2^2$, then $c = d^2$. Replacing x and y by x/d and dy (resp.), one may assume that $c = d = 1$. Now let

$$X = (1 + x + xy)^{-1}, \qquad Y = s_2(X) = xX, \qquad Z = s_2^2(X) = xyX.$$

Then $K = k(X, Y, Z)$, $X + Y + Z = 1$ and

$$s_1: X \to Z \to X, \qquad Y \to Y,$$
$$s_2: X \to Y \to Z \to X.$$

Hence K^G is generated by the elementary symmetric polynomials in X, Y, Z. Since $X + Y + Z = 1$, then $K^G = k(XY + YZ + ZX, XYZ)$ and hence rational.

Case 4. If $p(G) = G_4$, then one may assume that G acts as follows:

$$s_1: x \to y \to x,$$
$$s_2: x \to cy, \qquad y \to 1/dxy.$$

Since $s_1 s_2 s_1 = s_2^2$, then $c = 1$. Letting $z = 1/dxy$, one sees that K^G is generated by the elementary symmetric polynomials in x, y, z. Since $xyz = d^{-1} \in k$, then $K^G = k(x + y + z, xy + yz + zx)$ and hence rational.

Case 5. If $p(G) = G_5$, then one assumes that G acts as follows:

$$s_1: \quad x \to 1/y, \qquad y \to 1/x,$$
$$s_2: \quad x \to y/c, \qquad y \to 1/dx.$$

Since $s_1 s_2 s_1 = s_2^3$, then $d^2 = 1$. Replacing y by y/c, one sees that

$$s_1: \quad x \to 1/cy, \qquad y \to 1/cx,$$
$$s_2: \quad x \to y \to 1/dcx \to 1/dcy \to x.$$

If $d = 1$, let $X = x + 1/cx$, $Y = y + 1/cy$ and $H = \langle s_2^2, s_1 s_2 \rangle$. Then $k(X, Y) \subseteq K^H$ and $[K: k(X, Y)] = 4 = \text{order}(H)$. Hence $K^H = k(X, Y)$. Also, s_1 acts on $k(X, Y)$ as follows:

$$s_1: \quad X \to Y \to X.$$

Hence $K^G = (K^H)^{s_1}$ is rational.

If $d = -1$ (and $\text{char}(k) \neq 2$), let $\bar{k}$ be the splitting field over k of $T^2 + c = (T - r)(T + r)$ and let $\bar{K} = \bar{k}(x, y)$. Let $\Gamma = \text{Gal}(\bar{k}/k)$. Then Γ is either trivial or generated by

$$\gamma: \quad r \to -r.$$

Let $X = rx$, $Y = ry$ and for any $t \neq -1$, let $t' = (1 - t)/(1 + t)$. Then

$$s_1: \quad X \to -1/Y, \qquad Y \to -1/X,$$
$$s_2: \quad X \to Y \to 1/X \to 1/Y \to X,$$
$$\gamma: \quad X \to -X, \qquad Y \to -Y.$$

Let $A = X'Y'$, $B = X'/Y'$, $U = A'^2$, $V = rB'$. Then $\bar{K} = \bar{k}(x, y) = \bar{k}(X, Y) = \bar{k}(X', Y')$ and $s_2^2(X')/X' = s_2^2(Y')/Y' = -1$. Therefore $(\bar{K})^{s_2^2} = \bar{k}(A, B) = \bar{k}(A', B')$ and

$$s_1\colon\ A' \to -A', \qquad B' \to B',$$
$$s_2\colon\ A' \to 1/A', \qquad B' \to -1/B'.$$

Hence $(\bar{K})^{\langle s_2^2, s_1\rangle} = \bar{k}(U, V)$ and s_2 acts monomially, namely

$$s_2\colon\ U \to 1/U, \qquad V \to c/V.$$

Also, one sees that $\gamma(U) = U$ and $\gamma(V) = V$. Actually γ acts as follows:

	X	Y	X'	Y'	A	B	A'	B'	U	V
γ	$-X$	$-Y$	$1/X'$	$1/Y'$	$1/A$	$1/B$	$-A'$	$-B'$	U	V

Therefore $K^{\langle s_2^2, s_1\rangle} = k(U, V)$. Since s_2 acts monomially on $k(U, V)$, then $K^G = k(U, V)^{s_2}$ is rational by [2, Theorem 4].

Case 6. If $p(G) = G_6$, then one assumes that G acts as follows:

$$s_1\colon\ x \to y \to x,$$
$$s_2\colon\ x \to 1/cy, \qquad y \to dxy.$$

Since $s_1 s_2 s_1 = s_2^{-1}$, then $c = d^2$. Replacing x and y by dx and dy (resp.), one may assume that

$$s_1\colon\ x \to y \to x,$$
$$s_2\colon\ x \to 1/y, \qquad y \to xy.$$

Hence s_2^4: $x \to y \to 1/xy \to x$. Let $H = \langle s_1, s_2^4\rangle$ and $z = 1/xy$. Then K^H is generated by the elementary symmetric polynomials in x, y, z. Since $xyz = 1$, then $K^H = k(A, B)$, where

$$A = x + y + z = x + y + 1/xy,$$
$$B = xy + yz + zx = xy + 1/x + 1/y.$$

Also $s_2(A) = B$ and $s_2(B) = A$. Therefore $K^G = (K^H)^{s_2}$ is rational.

This completes the proof of the theorem. ∎

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